SUPG STABILIZATION PARAMETERS CALCULATED FROM THE QUADRATURE-POINT COMPONENTS OF THE ELEMENT-LEVEL MATRICES

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Abstract. We investigate, for linear and higher-order elements, various ways of calculating the advective limit of the stabilization parameter (“\( \tau \)”) used in the streamline-upwind/Petrov-Galerkin (SUPG) formulation of flow problems. In the context of a pure advection test problem, we compare the “UGN-based” and element-matrix-based calculations and the calculations from the quadrature-point components of the element-level matrices. Our investigation shows that the performances of all three definitions are comparable, with the element-based definitions yielding somewhat lower \( \tau \) values and the quadrature-point-based definitions somewhat higher. We also show that for both the element-based and quadrature-point-based definitions, as the polynomial orders increase, the \( \tau \) values decrease, as they should.
1 INTRODUCTION

Streamline-upwind/Petrov-Galerkin (SUPG) formulation for incompressible flows [1], SUPG formulation for compressible flows [2], and pressure-stabilizing/Petrov-Galerkin (PSPG) formulation [3] for incompressible flows are some of the most widely used stabilized formulations in finite element computation of flow problems. These formulations include a stabilization parameter that is mostly called “τ”. Typically this parameter involves a measure of the local length scale (i.e. the “element length”) and the local Reynolds and Courant numbers. Element lengths and τs were proposed for the SUPG formulation of incompressible and compressible flows in [1] and [2], respectively. These were followed by definitions introduced in [4], which we call here “UGN-based” definitions. More element lengths and τs were prescribed for the SUPG-based methods reported later. Additional τs, dependent upon spatial and temporal discretizations, were introduced in [5]. Later, τs applicable to higher-order elements were proposed in [6].

Calculating the τs based on the element-level matrices and vectors were introduced in [7]. These definitions are expressed in terms of the ratios of the norms of the matrices or vectors, and automatically take into account the local length scales, advection field and the element-level Reynolds and Courant numbers. Based on these definitions, a τ can be calculated for each element, or for each element node or degree of freedom or element equation, or, as it was proposed in [8, 9], for each quadrature point of each element.

A comparative investigation of the stabilization parameters and element lengths defined in [4, 7], and the element lengths defined in [1, 10], was carried out in [11]. These comparisons included calculations of the advective limit of the τ for higher-order elements, where, for various element types and shapes, variation of τ with the advection direction in an element was investigated. In [12], in the context of the actual computation of a pure advection test problem, a comparative investigation of the UGN-based, element-based, and element-node-based definitions of the advective limit of the τ was carried out.

Compared to an element-based τ, which is constant over the element, an element-node-based τ seemed to offer the advantage of letting τ vary over the element. However, it was reported in [12] that these definitions yield largely varying values and create difficulties in evaluating the τ at quadrature points in higher-order elements. In this study, we investigate the performances of the quadrature-point-based τ definitions proposed in [8, 9]. These τ definitions are expressed, for each quadrature point of each element, in terms of the ratios of the norms of the quadrature-point components of the element-level matrices. This is done with very little extra cost because those quadrature-point components need to be calculated anyway while forming the element-level matrices.

In Section 2, we describe the stabilized formulation we use for an advection-diffusion equation, and provide the UGN-based, element-based, and quadrature-point-based definitions of the advective limit of the τ. In Section 3, we report results from our comparative investigation in the context of a pure advection test problem. We present our concluding remarks in Section 4.
2 FORMULATION AND STABILIZATION PARAMETERS

2.1 Advection–diffusion equation and stabilized formulation

Consider over a domain Ω with boundary Γ the following time-dependent advection-diffusion equation, written on Ω and ∀t ∈ (0, T) as

$$ \frac{\partial \phi}{\partial t} + u \cdot \nabla \phi - \nabla \cdot (\nu \nabla \phi) = 0, $$ (1)

where φ represents the transported quantity, u is a divergence-free advection field, and ν is the diffusivity. The essential and natural boundary conditions associated with Eq. (1) are

$$ \phi = g \quad \text{on } \Gamma_g, \quad n \cdot \nu \nabla \phi = h \quad \text{on } \Gamma_h, $$ (2)

where Γ_g and Γ_h are complementary subsets of the boundary Γ, n is the unit normal vector, and g and h are given functions. A function φ_0(x) is specified as the initial condition.

Given suitably-defined finite-dimensional trial solution and test function spaces S\_φ and V\_φ, the stabilized finite element formulation of Eq. (1) can be written as follows: find φ\_h ∈ S\_φ such that ∀w\_h ∈ V\_φ:

$$ \int_{\Omega} w\_h \left( \frac{\partial \phi\_h}{\partial t} + u\_h \cdot \nabla \phi\_h \right) d\Omega + \int_{\Omega} \nabla w\_h \cdot \nu \nabla \phi\_h d\Omega - \int_{\Gamma_h} w\_h h d\Gamma + \sum_{e=1}^{n_{el}} \int_{\Omega^e} \tau_{SUPG} u\_h \cdot \nabla w\_h \left( \frac{\partial \phi\_h}{\partial t} + u\_h \cdot \nabla \phi\_h - \nabla \cdot (\nu \nabla \phi\_h) \right) d\Omega = 0. $$ (3)

Here n_{el} is the number of elements, Ω^e is the domain for element e, and τ_{SUPG} is the SUPG stabilization parameter.

2.2 UGN/RGN-based stabilization parameters

In this subsection, we describe the versions of the stabilization parameters (τs) denoted by the subscript UGN, namely the “UGN/RGN-based” stabilization parameters [13, 14]. For this purpose, we first define the unit vectors s and r:

$$ s = \frac{u\_h}{\|u\_h\|}, \quad r = \frac{\nabla |\phi\_h|}{\|\nabla |\phi\_h|\|}. $$ (4)

We define the components of (τ_{SUPG})\_UGN corresponding to the advection-, transient- and diffusion-dominated limits as given in [13, 14]:

$$ \tau_{SUGN1} = \left( \sum_{e=1}^{n_{el}} |u\_h \cdot \nabla N_e| \right)^{-1}, $$ (5)

$$ \tau_{SUGN2} = \frac{\Delta t}{2}, $$ (6)

$$ \tau_{SUGN3} = \frac{h_{RGN}^2}{4\nu}, $$ (7)
where \( n_{en} \) is the number of element nodes and \( N_a \) is the interpolation function associated with node \( a \), and the “element length” \( h_{RGN} \) is defined as

\[
h_{RGN} = 2 \left( \sum_{a=1}^{n_{en}} |r \cdot \nabla N_a| \right)^{-1}.
\] (8)

Based on Eq. (5), we define the “element length” \( h_{UGN} \) as

\[
h_{UGN} = 2 \| u^h \| \tau_{SUGN1}.
\] (9)

Although writing a direct expression for \( \tau_{SUGN} \) as given by Eq. (5) was pointed out in [15, 14, 8], the element length definition one obtains by combining Eq. (5) and Eq. (9) was first introduced (as a direct expression for \( h_{UGN} \)) in [4]. The expression for \( h_{RGN} \) as given by Eq. (8) was first introduced in [16]. We note that \( h_{UGN} \) and \( h_{RGN} \) can be viewed as the local length scales corresponding to the advection- and diffusion-dominated limits, respectively.

We now define \( (\tau_{SUPG})_{UGN} \) as follows:

\[
(\tau_{SUPG})_{UGN} = \left( \frac{1}{\tau_{SUGN1}^r} + \frac{1}{\tau_{SUGN2}^r} + \frac{1}{\tau_{SUGN3}^r} \right)^{-\frac{1}{r}}.
\] (10)

Eq. (10) is based on the inverse of \( (\tau_{SUPG})_{UGN} \) being defined as the \( r \)-norm of the vector with components \( \frac{1}{\tau_{SUGN1}}, \frac{1}{\tau_{SUGN2}} \) and \( \frac{1}{\tau_{SUGN3}} \). We note that the higher the integer \( r \) is, the sharper the switching between \( \tau_{SUGN1}, \tau_{SUGN2} \) and \( \tau_{SUGN3} \) becomes. This “\( r \)-switch” was introduced in [7]. Typically, we set \( r = 2 \). The expression for \( \tau_{SUGN3} \) given by Eq. (7) was proposed in [15, 14, 8].

## 2.3 Element-matrix-based stabilization parameters

With the notation \( \mathbf{b} : \int_{\Omega^e} (\ldots) d\Omega \) denoting the element-level matrix \( \mathbf{b} \) corresponding to the element-level integral \( \int_{\Omega^e} (\ldots) d\Omega \), the element-level matrices are defined as follows:

\[
\mathbf{m} : \int_{\Omega^e} w^h \frac{\partial \phi^h}{\partial t} d\Omega, \quad (11)
\]

\[
\mathbf{c} : \int_{\Omega^e} w^h u^h \cdot \nabla \phi^h d\Omega, \quad (12)
\]

\[
\mathbf{k} : \int_{\Omega^e} \nabla w^h \cdot \nu \nabla \phi^h d\Omega, \quad (13)
\]

\[
\mathbf{\bar{k}} : \int_{\Omega^e} u^h \cdot \nabla w^h \cdot u^h \cdot \nabla \phi^h d\Omega, \quad (14)
\]

\[
\mathbf{\bar{c}} : \int_{\Omega^e} u^h \cdot \nabla w^h \frac{\partial \phi^h}{\partial t} d\Omega. \quad (15)
\]
From [7], the element-level Reynolds and Courant numbers can be written as

\[ Re = \frac{\| u^b \|^2 \| c \|}{\nu \| k \|}, \quad (16) \]

\[ Cr_u = \frac{\Delta t \| c \|}{2 \| m \|}, \quad (17) \]

where \( \| b \| \) is the norm of matrix \( b \). Also from [7], we write the components of the element-matrix-based \( \tau_{\text{SUPG}} \):

\[ \tau_{S1} = \frac{\| c \|}{\| k \|}, \quad (18) \]

\[ \tau_{S2} = \frac{\Delta t \| c \|}{2 \| c \|}, \quad (19) \]

\[ \tau_{S3} = \frac{\tau_{S1} \| c \|}{\nu \| k_r \|} \quad \text{OR} \]

\[ = \tau_{S1} Re \left( \frac{h_{\text{RGN}}}{h_{\text{UGN}}} \right)^2, \quad (21) \]

and the construction of \( \tau_{\text{SUPG}} \):

\[ \tau_{\text{SUPG}} = \left( \frac{1}{\tau_{S1}'} + \frac{1}{\tau_{S2}'} + \frac{1}{\tau_{S3}'} \right)^{-\frac{1}{2}}. \quad (22) \]

We note that \( \tau_{S1}, \tau_{S2}, \text{and} \tau_{S3} \) are the limiting values for, respectively, the advection-dominated, transient-dominated, and diffusion-dominated cases. The definitions given by Eqs. (20) and (21) for the diffusion-dominated limit of the SUPG stabilization parameter were introduced in [17, 14, 8].

It was pointed out in [8, 9] that we can calculate a separate \( \tau \) for each quadrature point by using for that quadrature point the ratios of the norms of the element matrices or vectors contributed by that quadrature point. For example, a separate \( \tau_{S1} \) for each element quadrature point \( l \) would be calculated by using the expression

\[ (\tau_{S1})_l = \frac{\| c_l \|}{\| k_l \|}, \quad l = 1, 2, \ldots, n_{\text{int}}. \quad (23) \]

Here \( n_{\text{int}} \) is the number of quadrature points, and \( c_l \) and \( k_l \) are the element matrices contributed by the quadrature point \( l \).
3 TEST COMPUTATIONS

3.1 Test conditions

The test problem is (nearly) pure advection of a scalar \( \phi \) over a \( 1.0 \times 1.0 \) domain. The velocity field is rotational and counter-clock-wise. The velocity magnitude is proportional to the radial distance from the center of the domain located at \((x, y) = (0.0, 0.0)\). The maximum magnitude of the velocity, occurring at the domain corners, is \((0.50)^{1/2}\). Along the internal line from \((0.0, 0.0)\) to \((0, -0.5)\), the value of \( \phi \) is specified to be a cosine curve. For pure advection, the exact solution is a surface of revolution obtained by revolving this cosine curve around the \( z \)-axis.

Six different uniform meshes are employed: \( 24 \times 24 \) quadrilateral 4-noded elements with bilinear polynomials (Q4), \( 12 \times 12 \) quadrilateral 9-noded elements with biquadratic polynomials (Q9), \( 8 \times 8 \) quadrilateral 16-noded elements with bicubic polynomials (Q16), \( 24 \times 24 \times 2 \) triangular 3-noded elements with linear polynomials (T3), \( 12 \times 12 \times 2 \) triangular 6-noded elements with quadratic polynomials (T6), and, \( 8 \times 8 \times 2 \) triangular 10-noded elements with cubic polynomials (T10). The meshes with triangular elements are generated from the meshes with quadrilateral elements by dividing each quadrilateral along its diagonal in the direction \( 45^\circ \) from the \( x \)-axis. With this type of mesh designs, for the purpose of comparison, solutions obtained with all meshes can be projected onto the mesh with Q4 elements. The diffusivity is set to \( 1.0 \times 10^{-8} \), which leads to a maximum element Peclet number of \( 1.473 \times 10^{6} \) (based on the maximum velocity magnitude and the nodal spacing for the mesh with Q4 elements). This is a good approximation to a pure advection case. In these test computations, the element-based \( \tau_{s1} \) and quadrature-point-based \( \tau_{s1} \) are calculated with a norm definition that consists of taking the square root of the sum of the terms squared.

3.2 Test results

An elevation surface for a typical solution, obtained with the Q4 elements and element-based \( \tau_{s1} \), is shown in Figure 1. To more clearly show the effects of selecting different \( \tau \) definitions, the solutions will be displayed in the constant-\( x \) and constant-\( y \) planes intersecting at the point indicated in Figure 1 by an arrow. The errors in the solutions are more noticeable in the constant-\( y \) plane, and therefore the majority of our planar solution displays will be in that plane. Before we compare the actual solutions obtained with different \( \tau \) definitions, we first compare, with contour plots, the \( \tau \) values calculated from those different definitions. Having several different choices for \( \tau \) and element type makes it difficult to have a single contour range in displaying the variation of the \( \tau \) values over the domain. The contour values we report here start from 0.2 and increase in steps of 0.2. The exception is for the cubic elements with the element-based \( \tau_{s1} \), where the values are smaller than 0.15 and therefore only the 0.1 contour is shown. Due to the rotational velocity field, all of the computed \( \tau \) values have four symmetry planes for the quadrilateral elements and two for the triangular elements. The surface fitting algorithm
of the contour utility did not always truly represent the symmetries present.

Figures 2 and 3 show, for the meshes with Q4 and T3 elements, respectively, the contours of $\tau$ values for $\tau_{\text{SGN1}}$ and element-based and quadrature-point-based $\tau_{S1}$. Figures 4 and 5 show, for the meshes with quadrilateral elements, the quadrature-point-based and element-based $\tau_{S1}$ values, respectively. The quadrature-point-based and element-based $\tau_{S1}$ values, for the meshes with triangular elements, are shown in Figures 6 and 7, respectively. From these contour plots, we observe that, for both quadrature-point-based and element-based $\tau_{S1}$, as the polynomial orders increase, the $\tau$ values decrease, and that is how it should be. In Figures 6 one can see an effect of the mesh bias caused by using a 45° side on every triangular element. When the local velocity vector is tangent to that longest element side the quadrature point nearest that side yields a relatively large $\tau$ value, as would be expected. A similar behavior has also been observed in other test problems having uniform velocity fields.

We also compare the actual solutions obtained with different $\tau$ definitions by displaying those solutions in the constant-$y$ and constant-$x$ planes we defined in Figure 1. Figure 8 shows, for the meshes with Q4 and T3 elements, the solutions in the constant-$y$ plane, obtained with the $\tau_{\text{SGN1}}$, element-based $\tau_{S1}$, and quadrature-point-based $\tau_{S1}$ definitions. Figure 9 shows, for the meshes with Q4, Q9, and Q16 elements, the solutions in the
constant-$y$ plane, obtained with the quadrature-point-based and element-based $\tau_{S1}$ definitions. For the T3, T6, and T10 elements, the solutions in the constant-$y$ plane, obtained with the quadrature-point-based and element-based $\tau_{S1}$ definitions, are shown in Figure 10. We observe that, as expected for a problem with smooth solution, the accuracy increases with the increase in the polynomial order. We also observe that increases in accuracy are not as noticeable with upgrade from quadratics to cubics as they are with upgrade from linears to quadratics. Solutions in the constant-$x$ plane, obtained with the quadrature-point-based $\tau_{S1}$ (S1 QP) definition, are shown in Figure 11 for Q4, Q9, and Q16 elements as well as T3, T6, and T10 elements. Because of the scale of these plots, the differences in the solutions obtained with different meshes and different $\tau$ definitions are less noticeable. Therefore we do not repeat this display for the solutions obtained with the other $\tau$ definition.

4 CONCLUDING REMARKS

We provided a comparative investigation of various ways of calculating the advective limit of the stabilization parameter (“$\tau$”) used in the streamline-upwind/Petrov-Galerkin (SUPG) formulation. The investigation was carried out in the context of the actual computation of a pure advection test problem with linear and higher-order triangular and quadrilateral elements. We focused our attention to the UGN-based, element-based, and quadrature-point-based definitions of the $\tau$. We compared how $\tau$ values calculated based on different definitions vary in the computational domain. We also compared the qualities of the solutions obtained with the different definitions. The test computations showed that the quadrature-point-based definitions yield higher $\tau$ values in local regions where the velocity vector is rapidly changing its magnitude and direction. The test computations also showed that for both the quadrature-point-based and element-based definitions, as expected, the $\tau$ values decrease as the polynomial orders increase, though to a lesser extent for the quadrature-point-based definitions. The element-based definition, in addition to providing still a general framework that automatically takes into account the local length scales and the advection field, yields smoother and more conservative $\tau$ values. These features make it our favorite definition.

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Figure 2: Contours of $\tau$ values for the mesh with Q4 elements: $\tau_{\text{SUGN1}}$ (top), element-based $\tau_{S1}$ (lower left), and quadrature-point-based $\tau_{S1}$ (lower right).
Figure 3: Contours of $\tau$ values for the mesh with T3 elements: $\tau_{\text{SUGN1}}$ (top), element-based $\tau_{S1}$ (lower left), and quadrature-point-based $\tau_{S1}$ (lower right).
Figure 4: Contours of quadrature-point-based $\tau_{s1}$ values for meshes with quadrilateral elements: Q4 (top), Q9 (lower left), and Q16 (lower right).
Figure 5: Contours of element-based $\tau_{S1}$ values for meshes with quadrilateral elements: Q4 (top), Q9 (lower left), and Q16 (lower right).
Figure 6: Contours quadrature-point-based $\tau_{S1}$ values for meshes with triangular elements: T3 (top), T6 (lower left), and T10 (lower right).
Figure 7: Contours of element-based $\tau_{S1}$ values for meshes with triangular elements: T3 (top), T6 (lower left), and T10 (lower right).
Figure 8: Solutions in constant-$y$ plane, for meshes with Q4 (top) and T3 (bottom) elements, obtained with $\tau_{S1}$ (SUGN1), element-based $\tau_{S1}$ (S1 E), and quadrature-point-based $\tau_{S1}$ (S1 QP) definitions.
Figure 9: Solutions in constant-\(y\) plane, for meshes with Q4, Q9, and Q16 elements, obtained with quadrature-point-based \(\tau_{S1}\) (S1 QP) and element-based \(\tau_{S1}\) (S1 E) definitions.
Figure 10: Solutions in constant-\(y\) plane, for meshes with T3, T6, and T10 elements, obtained with quadrature-point-based \(\tau_{S1} (S1 \text{ QP})\) and element-based \(\tau_{S1} (S1 \text{ E})\) definitions.
Figure 11: Solutions in constant-$x$ plane, for meshes with Q4, Q9, and Q16 elements (top) and T3, T6, and T10 elements (bottom), obtained with quadrature-point-based $τ_{S1}$ (S1 QP) definition.